Math 601 Midterm 1 Sample

Name: ____________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	18	
2	10	
3	15	
4	10	
5	10	
6	6	
7	10	
8	5	
9	16	
Total:	100	

Question 1. (18 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) Let V be a linear subspace of $\mathbb{R}^n$. We have vectors $v_1, \cdots, v_k$ and $w_1, \cdots, w_\ell$ in V. Suppose $v_1, \cdots, v_k$ are linearly independent, and $w_1, \cdots, w_\ell$ span V. Then $k \leq \ell$.

(b) Let U and W be subspaces of the vector space V. If $U \subseteq W$, then $U + W = W$.

(c) An $(n \times n)$ matrix is invertible if and only if it is row equivalent to the $(n \times n)$ identity matrix I_n.

(d) Suppose A is an $(n \times m)$ matrix, then $\dim \operatorname{Im}(A) \leq n$ and $\dim \ker(A) \leq m$.

(e) Let A, B, C be three $(n \times n)$ square matrices. If $AB = AC$, then $B = C$.

(f) The linear system

$$\begin{array}{rcrcrcr} x & + & 10y & - & 3z & = & 3 \\ 3x & + & 4y & + & 9z & = & 1 \\ 2x & + & 5y & - & 2z & = & 8 \end{array}$$

has exactly two solutions.

Solution:

(a) True

(b) True

(c) True

(d) True

(e) False

(f) False (a linear system can never have precisely two solutions.)

Question 2. (10 pts)

A line L in $\mathbb{R}^3$ passes through the point $(0, 1, 0)$. Suppose L is parallel to the plane $x + y + z = 0$ and is orthogonal to the line

$$x = 2t, y = t + 1, z = -t.$$

Find parametric equations of the line L.

Solution: The direction vector of L is orthogonal to both the normal vector $u = (1, 1, 1)$ of the plane and the direction vector $v = (2, 1, -1)$ of the other line.

Calculate the cross product of u and v:

$$u \times v = (-2, -3, -1)$$

So we have the following parametric equations of L:

$$\begin{cases} x = -2t \\ y = 3t + 1 \\ z = -t \end{cases}$$

Question 3. (15 pts)

Given

$$A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$$

(a) Find a basis of $\mathrm{Ker}(A)$.

Solution: First, use elementary row operations to get the reduced row echelon form of A.

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So all elements in $\mathrm{Ker}A$ are of the form

$$t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

So

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

form a basis of the kernel.

(b) Find a basis of the row space of A.

Solution: The three nonzero rows in the reduced row echelon form of A form a basis of the row space of A. That is

$$u_1 = (1, 1, 1, 1, -1)$$

$$u_2 = (0, 0, -5, -1, 15)$$

$$u_3 = (0, 0, 0, 1, 5)$$

form a basis of the row space of A.

(c) Find a basis of $\text{Im}(A)$.

Solution: Use $\text{rref}(A)$ from the part (a), we see that the 1st, 3rd and 4th columns of A form a basis of $\text{Im}(A)$. That is,

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}, w_2 = \begin{bmatrix} -3 \\ 1 \\ -5 \\ -2 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

form a basis of $\text{Im}(A)$.

Question 4. (10 pts)
Let $M_2(\mathbb{R})$ be the space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the basis S.

Solution: We need to write

$$\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we need to solve the linear system

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = 5 \\ a_1 - a_2 - a_3 = 3 \\ a_1 + a_2 = 3 \\ a_1 = 1 \end{cases}$$

Simply use back substitution. We have

$$a_1 = 1, a_2 = 2, a_3 = -4, a_4 = 6$$

So

$$[A]_S = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 6 \end{bmatrix}$$

Question 5. (10 pts)

Determine whether $x = \begin{bmatrix} 4 \\ 5 \\ 6 \\ -1 \end{bmatrix}$ lies in the linear span of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: Write down the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 3 & 4 & -2 & 5 \\ 2 & -1 & 1 & 6 \\ 5 & 2 & 3 & -1 \end{array}\right]$$

by applying elementary row operations, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 55/12 \end{array}\right]$$

This is inconsistent. So no solution. In other words, x is not in the linear span of v_1, v_2 and v_3.

Question 6. (6 pts)

Let V be a vector space. Suppose $F : V \to V$ is a linear transformation. Show that the kernel of F is a subspace of V.

Solution:

(1) $0 \in \ker F$, since $F(0) = 0$.

(2) If $v, w \in \ker F$, then

$$F(v + w) = F(v) + F(w) = 0 + 0 = 0.$$

So $v + w \in \ker F$.

(3) If $w \in \ker F$ and $k \in \mathbb{R}$, then

$$F(kw) = kF(w) = k \cdot 0 = 0.$$

So $kw \in \ker F$.

Therefore $\ker F$ is a subspace of V.

Question 7. (10 pts)
Show that $(t-1)$, $(t+1)$ and $(t-1)^2$ form a basis of $\mathbb{P}_2(t)$, where $\mathbb{P}_2(t)$ is the space of all polynomials of degree ≤ 2.

Solution: There are various ways to solve this problem.

(1) First method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and span $\mathbb{P}_2(t)$.

(2) Second method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and use the fact $\dim \mathbb{P}_2(t) = 3$.

(3) third method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ span $\mathbb{P}_2(t)$ and use the fact $\dim \mathbb{P}_2(t) = 3$.

Let us the second method. Consider a linear combination of $(t-1)$, $(t+1)$ and $(t-1)^2$ such that

$$a_1(t-1) + a_2(t+1) + a_3(t-1)^2 = 0.$$

Then we want to show that $a_1 = a_2 = a_3 = 0$ is the unique solution. This would imply that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Regroup the coefficients, and we have the following linear system:

$$\begin{cases} -a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 - 2a_3 = 0 \\ a_3 = 0 \end{cases}$$

Solve this and indeed we have the unique solution $a_1 = a_2 = a_3 = 0$. So $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Now we know that $\dim \mathbb{P}_2(t) = 3$. Then any 3 linearly independent vectors of $\mathbb{P}_2(t)$ form a basis. Therefore $(t-1)$, $(t+1)$ and $(t-1)^2$ form a basis.

Question 8. (5 pts)

Suppose A and B are invertible $(n \times n)$ matrices. Then we know that AB is also invertible. Use this fact and the definition of the inverse of an invertible matrix to show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Solution: Notice that

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$$

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A^{-1}IA = I$$

So by definition the inverse of AB, which is denoted by $(AB)^{-1}$, is $B^{-1}A^{-1}$. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Question 9. (16 pts)

Recall that two vectors $v, w \in \mathbb{R}^n$ are said to be orthogonal if their dot product is zero, that is, $v \cdot w = 0$.

(a) Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. Determine whether u_1, u_2 and u_3 are mutually orthogonal.

Solution: A straightforward calculation shows that u_1, u_2 and u_3 are mutually orthogonal.

(b) Suppose v and w are both nonzero vectors in $\mathbb{R}^n$. Show that if $v \cdot w = 0$, then v and w are linearly independent. (Hint: For any linear combination of v and w, take the dot product of this linear combination with v (respectively w). What do you see?)

Solution: Suppose we have a linear combination

$$a_1 v + a_2 w = 0.$$

Then we need to show that $a_1 = a_2 = 0$.
Indeed, consider the dot product

$$0 = (a_1 v + a_2 w) \cdot v = a_1 \|v\|^2 + 0 = a_1 \|v\|^2$$

But $\|v\| \neq 0$. Therefore $a_1 = 0$.
Similarly,

$$0 = (a_1 v + a_2 w) \cdot w = 0 + a_2 \|w\|^2 + 0 = a_2 \|w\|^2$$

shows that $a_2 = 0$.
So v and w are linearly independent.

(c) Now suppose nonzero vectors v_1, v_2 and v_3 are mutually orthogonal in $\mathbb{R}^n$. Show that the set $\{v_1, v_2, v_3\}$ is linearly independent. (Hint: the same idea from part (b) applies.)

Solution: Suppose we have a linear combination

$$a_1v_1 + a_2v_2 + a_3v_3 = 0.$$

Then we need to show that $a_1 = a_2 == a_3 = 0$.
Consider the dot product

$$0 = (a_1v_1 + a_2v_2 + a_3v_3) \cdot v_1 = a_1\|v_1\|^2$$

But $\|v_1\| \neq 0$. Therefore $a_2 = 0$.
Similarly, by taking the dot product of $a_1v_1 + a_2v_2 + a_3v_3$ with v_2 and v_3 respectively, we have $a_2 = a_3 = 0$.
Therefore, the set $\{v_1, v_2, v_3\}$ is linearly independent.

(d) Use either the previous parts or your other favorite method to determine whether $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

Solution: From part (a), we know that u_1u_2 and u_3 are mutually orthogonal and are nonzero vectors.
By part (c), we know u_1, u_2 and u_3 are linearly independent.
Since $\dim \mathbb{R}^3 = 3$, we see that u_1, u_2 and u_3 form a basis of $\mathbb{R}^3$.